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Fourier Series Solution of Second Order Ordinary Differential Equations Arising from Spring-Mass System

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Abstract: **Fourier series have important application in ordinary differential equations evolving from practical applications of Physics and engineering. The coefficients of Fourier series were obtained by means of integration from the series. Thereafter, they were applied to spring-mass systems which were acted on by a forcing function. It was demonstrated with examples that it is possible to obtain the transient and steady-state solution of any given mass-spring problem when acted upon by an external force to produce oscillation.**

Keywords: **Fourier Series, spring-mass problem, oscillation, ordinary differential equation, periodic function, orthogonal.**

I. INTRODUCTION

Real life phenomena such as heart beat, the motion of electrical charges, the movement of a pendulum and both rotation and revolution of the earth are typical examples of phenomena that repeats at regular intervals. Thus, the mathematical expression of these phenomena as function f repeating at regular intervals of the independent variable are known as periodic functions. These functions are best treated with Fourier series and Fourier transforms.

Fourier series is a representation of infinite series that are designed to represent general periodic functions by means of simpler ones (sines and cosines). The orthogonality of trigonometric functions enables the computation of the coefficients of the Fourier series with by means of Euler formulas.

Fourier series play vital role in in the solution of ODEs resulting from damped oscillations. Similarly, Fourier series find application in PDEs as some discontinuous periodic functions do not have Taylor series representation and are best handled by Fourier series. [1]

Definition: A function $f(x)$ is said to be periodic with period p if there exists a positive number p such that

$f(x+p) = f(x)$ [1], [6], [7].

In this work we explore the development of Fourier series and apply it spring-mass oscillations with forcing function.

II. THE FOURIER SERIES

Consider the series of a periodic function trigonometrically expressed as

$$a_0 + a_1 cos(x) + b_1 sin(x) + a_2 cos(2x) + b_2 sin(2x) + a_3 cos(3x) + b_3 sin(3x) + \cdots \quad (1)$$

Then (1) can be expressed as

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi t}{L}\right) + b_n sin\left(\frac{n\pi t}{L}\right)\right) \quad (2)$$

where a_0, a_n and b_n are coefficients to be determined.

Integrating (2) we have

$$\int_{-L}^{L} f(t)\,dx = a_0 \int_{-L}^{L} dt + \sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L} cos\left(\frac{n\pi t}{L}\right) dt + b_n \int_{-L}^{L} sin\left(\frac{n\pi t}{L}\right) dt\right) \quad (3)$$

Since $cos\left(\frac{n\pi t}{L}\right)$ and $sin\left(\frac{n\pi t}{L}\right)$ are orthogonal for $n\epsilon Z^{+}$on the interval $(-L, L)$ (**) becomes

$\int_{-L}^{L} f(t)\,dx = a_0 x|_{-L}^{L} = 2La_0$ hence

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dx \quad (4)$$

Multiplying (*) by $cos\left(\frac{m\pi t}{L}\right)$ and integrating gives

$$\int_{-L}^{L} f(t)\, cos\left(\frac{m\pi t}{L}\right) dx = a_0 \int_{-L}^{L} cos\left(\frac{m\pi t}{L}\right) dt$$
$$+\sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L} cos\left(\frac{m\pi t}{L}\right) cos\left(\frac{n\pi t}{L}\right) dt + b_n \int_{-L}^{L} cos\left(\frac{m\pi t}{L}\right) sin\left(\frac{n\pi t}{L}\right) dt\right) \quad (5)$$

Using the conditions

$$\begin{cases} \int_{-L}^{L} cos\left(\frac{m\pi t}{L}\right) dt = 0, m > 0 \\ \int_{-L}^{L} sin\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt = 0 \\ \int_{-L}^{L} cos\left(\frac{m\pi t}{L}\right) cos\left(\frac{n\pi t}{L}\right) dt = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases} \end{cases} \quad (6)$$

So that from (6)

$$\int_{-L}^{L} f(t)\, cos\left(\frac{n\pi t}{L}\right) dx = \sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L} cos\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt + b_n \int_{-L}^{L} sin\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt\right)$$

$$\int_{-L}^{L} f(t)\, cos\left(\frac{m\pi t}{L}\right) dx = a_0 \int_{-L}^{L} cos\left(\frac{m\pi t}{L}\right) dt$$
$$+\sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L} cos\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt + b_n \int_{-L}^{L} sin\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt\right)$$

$$\int_{-L}^{L} f(t)\, cos\left(\frac{n\pi t}{L}\right) dt = a_n L,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\, cos\left(\frac{n\pi t}{L}\right) dt \quad (7)$$

Similarly, Multiplying (2) by $sin\left(\frac{m\pi t}{L}\right)$ and integrating gives

$$\int_{-L}^{L} f(t)\, sin\left(\frac{m\pi t}{L}\right) dx = a_0 \int_{-L}^{L} sin\left(\frac{m\pi t}{L}\right) dt$$
$$+\sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L} sin\left(\frac{m\pi t}{L}\right) cos\left(\frac{n\pi t}{L}\right) dt + b_n \int_{-L}^{L} sin\left(\frac{m\pi t}{L}\right) sin\left(\frac{n\pi t}{L}\right) dt\right) \quad (8)$$

Using the conditions

$$\begin{cases} \int_{-L}^{L} sin\left(\frac{m\pi t}{L}\right) dt = 0, m > 0 \\ \int_{-L}^{L} sin\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt = 0 \\ \int_{-L}^{L} sin\left(\frac{m\pi t}{L}\right) sin\left(\frac{n\pi t}{L}\right) dt = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases} \end{cases} \quad (9)$$

we obtain

$$\int_{-L}^{L} f(t)\, sin\left(\frac{n\pi t}{L}\right) dt = \sum_{n=1}^{\infty}\left(b_n \int_{-L}^{L} sin\left(\frac{n\pi t}{L}\right) cos\left(\frac{m\pi t}{L}\right) dt\right)$$

$$\int_{-L}^{L} f(t)\, sin\left(\frac{n\pi t}{L}\right) dt = a_0 \int_{-L}^{L} sin\left(\frac{m\pi t}{L}\right) dt$$

$\int_{-L}^{L} f(t)\, sin\left(\frac{n\pi t}{L}\right) dt = b_n L,$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\, sin\left(\frac{n\pi t}{L}\right) dt \qquad (10)$$

Hence (4), (7) and (10) give the coefficients of the Fourier series

$$\begin{cases} a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt \\ a_n = \frac{1}{L}\int_{-L}^{L} f(t)\, cos\left(\frac{nt\pi}{L}\right) dx, \ n = 1,2,3,\cdots \\ b_n = \frac{1}{L}\int_{-L}^{L} f(t)\, sin\left(\frac{nt\pi}{L}\right) dt, \ n = 1,2,3,\cdots \end{cases} \quad [8], [9].$$

III. THE SPRING-MASS PROBLEM

Consider a spring-mass system with mass m. If an external force, or the forcing function, $F(t)$ acts on the system, causing a displacement x units from equilibrium and k the spring constant, then by Newton's Second Law, we obtain the second order Ordinary Differential Equation (ODE)

$$mx'' + \gamma x' + kx = F(t) \qquad (11)$$

With the initial conditions

$x(0) = x_0$ initial displacement from equilibrium position

$x'(0) = x'_0$ initial velocity

whose natural frequency ω_0 is defined as $\omega_0 = \sqrt{k/m}$, a non-homogenous second order ODE where x is the displacement from equilibrium and $F(t)$ is the applied external force in a mass-spring system [2], [3].

The solution of the above equation will be of the form $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_P(t)$ which can be succinctly written as

$$y(t) = y_c(t) + y_P(t) \qquad (12)$$

where $y_c(t)$ and $y_P(t)$ are the transient and steady-state solution respectively.

and constants c_1 and c_2 are determined by the initial condition of (12) [6].

$F(t)$ and $y(t)$ are the input and response respectively. If for all values of c_1 and c_1,

$c_1 y_1(t) + c_2 y_2(t) \to 0$ as $t \to \infty$, the system (2) is said to be stable and $y_P(t)$ is called the steady state solution of (12).

Since the system (11) is such that the auxiliary equation will be of the form $r^2 = -k$, r will have complex roots, yielding the general solution

$$y(t) = c_1 cos(\omega_0) + c_2 sin(\omega_0) + y_P(t) \qquad (13)$$

where $y_c(t) = c_1 cos(\omega_0) + c_2 sin(\omega_0)$ is the complimentary function and $y_P(t)$, the particular solution of (1)

$y_P(t)$ can be obtained by defining it in terms of Fourier series as

$$y_P(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{nt\pi}{L}\right) + b_n sin\left(\frac{nt\pi}{L}\right)\right) \qquad (14)$$

where $F(t)$ in (1) has period $2L$ and is assumed to be piecewise smooth [3], [4], [5].

If we assume $L = \pi$, (1) becomes

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n cos(nt) + b_n sin(nt)\right) \qquad (15)$$

and (3)-(5) are defined as

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt \qquad (16)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, cosntdt, \ n = 1,2,3,\cdots \qquad (17)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, sindt, \ n = 1,2,3,\cdots \qquad (18)$$

IV. APPLICATION

We hereby apply Fourier series to obtain forcing function $F(t)$ In the first example, the forcing function is periodic while in the second example, it is non-periodic.

Example 1:

Let $F(t)$ be an external force causing a displacement x from equilibrium of an undamped spring-mass system yielding the ODE $x''(t)+9x(t)=F(t), x(0)=0,\ x'(0)=1$. Find the general solution of the ODE where $F(t)$ is defined as

$$x''(t)+9x(t)=F(t),\quad F(t)=\begin{cases}0, & -\pi<t<-\frac{\pi}{2}\\ 4, & -\frac{\pi}{2}<t<\frac{\pi}{2}\\ 0, & \frac{\pi}{2}<t<\pi\end{cases}$$

To obtain the solution for y_{cf} let $x''(t)+9x(t)=0$, that is $F(t)=0$. The auxiliary equation will be $r^2+9=0, r^2=-9, r=\pm 3i$ giving the complementary function $x_c(t)=c_1\cos 3t+c_2\sin 3t$

$$x_c(0)=c_1\cos(0)+c_2\sin(0)=0,\ \ c_1=0,$$

$$x'_c(0)=-3c_1\sin(0)+3c_2\cos(0)=1,\ \ c_2=\frac{1}{3}$$

The general solution of the problem is

$$x(t)=c_1\cos 3t+c_2\sin 3t+x_p(t),$$

$$x(t)=\frac{1}{3}\sin 3t+x_p(t)$$

$$F(t)=\begin{cases}0, & -\pi<t<-\frac{\pi}{2}\\ 4, & -\frac{\pi}{2}<t<\frac{\pi}{2}\\ 0, & \frac{\pi}{2}<t<\pi\end{cases}$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt \qquad \frac{1}{2\pi}\left\{\int_{-\pi}^{-\frac{\pi}{2}}0dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4dt+\int_{\frac{\pi}{2}}^{\pi}0dx\right\}=4t|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=4$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)dt=\frac{1}{\pi}\left\{\int_{-\pi}^{-\frac{\pi}{2}}0\cos(nt)dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos(nt)dt+\int_{\frac{\pi}{2}}^{\pi}0\cos(nt)dt\right\}$$

$$=\frac{4}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(nt)dt=\frac{4}{n\pi}\sin(nt)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_n=\frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right)=\begin{cases}0, & n=2,4,6,\cdots\\ 1, & n=1,5,9,\cdots\\ -1, & n=3,7,11,\cdots\end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nt)dt=\frac{1}{\pi}\left\{\int_{-\pi}^{-\frac{\pi}{2}}0\sin(nt)dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\sin(nt)dt+\int_{\frac{\pi}{2}}^{\pi}0\sin(nt)dt\right\}$$

$$=\frac{4}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin(nt)dt=-\frac{4}{n\pi}\cos(nt)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=-\frac{4}{n\pi}\left\{\left(-\cos\left(\frac{n\pi}{2}\right)\right)-\left(-\cos\left(-\frac{n\pi}{2}\right)\right)\right\}$$

$b_n=0$ since cosine is even, that is, $\cos(-x)=\cos(x)$

$$x_p(t)=4+\frac{8}{\pi}\left\{\cos t-\frac{1}{3}\cos 3t+\frac{1}{5}\cos 5t-\frac{1}{7}\cos 7t+\cdots\right\}$$

Hence

$$x(t)=\frac{1}{3}\sin 3t+4+\frac{8}{\pi}\left\{\cos t-\frac{1}{3}\cos 3t+\frac{1}{5}\cos 5t-\frac{1}{7}\cos 7t+\cdots\right\}$$

Example 2:

A 10-kg mass is attached to a spring with a spring constant of 300 N/m is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(t) = \sqrt{2}t$ .Find the subsequent motion of the mass if the force due to air resistance is $-110x'N$

With $m = 10,\ k = 300,\ a = -110$ and $F(t) = \sqrt{2}t$ Applying (1) with its initial conditions gives

$$mx'' = -kx - ax' + F(t)$$

$$x'' + \frac{a}{m}x' + \frac{k}{m} = \frac{F(t)}{m}$$

$$x'' + 11x' + 30x = \sqrt{2}t.$$

To obtain the solution for y_c let $x''(t) + 11x' + 30x(t) = 0$, that is $F(t) = 0$. The auxiliary equation will be $r^2 + 11r + 20 = 0,\ \ r = -5, -7$ giving the complementary function

$$x_c(t) = c_1e^{-5x} + c_2e^{-7x}$$

Using the initial condition $x(0) = 0, x'^{(0)} = -1$

$$x_c(0) = c_1 + c_2 = 0,\ \ c_2 = -c_1$$

$$x'_c(0) = -5c_1 - 7c_2 = 1,\ \ c_1 = -\frac{1}{2},\ \ \frac{1}{2}$$

The general solution of the problem is

$$x_c(t) = -\frac{1}{2}e^{-5x} + \frac{1}{2}e^{-7x}$$

$$x(t) = -\frac{1}{2}e^{-5x} + \frac{1}{2}e^{-7x} + x_p(t)$$

$$F(x) = \sqrt{2}t,\quad -\pi < t < \pi$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{2}tdt \qquad = \frac{1}{2\pi}\frac{t^2}{2}\Big|_{-\pi}^{\pi} \qquad = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sqrt{2}t\cos(nt)dt$$

$$= \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} t\cos(nt)dt$$

Employing integration by parts, we obtain

$$a_n = \frac{\sqrt{2}}{\pi}\left(\frac{1}{\pi}\frac{\sin(nt)}{n} - \frac{\cos(nt)}{n^2}\Big|_{-\pi}^{\pi}\right) \qquad = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \frac{\sqrt{2}}{\pi}\left(-\frac{t\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{t\cos(nt)}{n}dt\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \frac{\sqrt{2}}{\pi}\left(-\frac{t\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} t\cos(nt)dt\right)$$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,sin(nt)dt = \frac{\sqrt{2}}{\pi}\left(-\frac{tcos(nt)}{n} + \frac{1}{n}\left(\frac{tsin(nt)}{n} + \frac{cos(nt)}{n^2}\right)\Big|_{-\pi}^{\pi}\right)$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,sin(nt)dt$

$= \frac{2\sqrt{2}(-1)^{n+1}}{\pi}$

$x_P(t) = \frac{2\sqrt{2}(-1)^{n+1}}{\pi}$

$x(t) = -\frac{1}{2}e^{-5x} + \frac{1}{2}e^{-7x} + x_p(t)$

$x(t) = \frac{1}{2}(-e^{-5x} + e^{-7x}) + \frac{2\sqrt{2}(-1)^{n+1}}{\pi}$

V. CONCLUSION

We have considered in this work the development of the Fourier coefficients which have application in differential equations. We have equally applied the series to obtain the forcing functions of two problems considered and expressed the entire problem as $y(t) = y_c(t) + y_P(t)$. We were able to express both forcing functions in terms of Fourier series which gives the complementary function as transient solution and the particular solution as the steady-state solution.

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